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The equivalence of Lagrange's equations of motion of the first kind and the generalized inverse form

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Abstract. It is shown that the generalized inverse form of the equations of motion is equivalent to Lagrange's equations of motion of the first kind.

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1. Introduction

The purpose of this paper is to show that the generalized inverse (GI) form of the equations of motion for discrete constrained mechanical systems is equivalent to Lagrange's equations of motion of the first kind. This equivalence leads to a simple proof of Gauss' principle of least constraint. It also shows that any singular behavior in the integration of the GI form is also present in the classical Lagrange form.

2. Lagrange's equations of motion of the first kind

Lagrange's equations of motion of the first kind take the form

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{a} + \boldsymbol{A}^T \boldsymbol{\lambda}, \\ \boldsymbol{A}\ddot{\boldsymbol{x}} = \boldsymbol{b}. \end{cases} \tag{1}$$

An excellent discussion is available in Sommerfeld's "Mechanics" (1952, pp. 66–69).

Recently (Kalaba & Udwadia 1992 and Udwadia & Kalaba 1992), it was discovered that,

$$\ddot{x} = a + M^{-1/2}(AM^{-1/2})^{+}(b - Aa), \tag{2}$$

which is the generalized inverse (GI) form of the equations of motion. In the next section, we show the equivalence between Eqs. (1) and (2).

3. Proof

To simplify matters, we first transform Eqs. (1) and (2) into the following equivalent forms:

$$\begin{cases} M^{1/2}\ddot{x} = M^{1/2}a + M^{-1/2}A^{T}\lambda, \\ AM^{-1/2}M^{1/2}\ddot{x} = b, \end{cases} \tag{3}$$

and

$$M^{1/2}\ddot{x} = M^{1/2}a + \left(AM^{-1/2}\right)^{+}\left(b - AM^{-1/2}M^{1/2}a\right). \tag{4}$$

Then, let $y = M^{1/2}\ddot{x}$, $g = M^{1/2}a$, and $C = AM^{-1/2}$. Since $M^{-1/2}$ is symmetric, it is seen that $C^{T} = \left(AM^{-1/2}\right)^{T} = M^{-1/2}A^{T}$. Therefore, Eqs. (3) and (4) are reduced to

$$\begin{cases} y = g + C^{T}\lambda, & (5) \\ Cy = b. & (6) \end{cases}$$

and

$$y = g + C^{+}(b - Cg). \tag{7}$$

In the following, we shall show that Eq. (7) is the existent and unique solution to the set of linear Eqs. (5)–(6).

We first prove the existence. That is, we would like to show that Eq. (7) satisfies Eqs. (5) and (6). Since the matrices C^{+} and C^{T} span the same column space, it follows that there exists a vector λ such that $C^{T}\lambda = C^{+}(b - Cg)$. This implies that Eq. (5) holds. Substituting Eq. (7) into the right hand side of Eq. (6) gives $Cy = Cg + CC^{+}(b - Cg) = Cg + CC^{+}b - Cg = CC^{+}b$. Since the consistency of Eq. (6) requires that $CC^{+}b = b$, the right hand side is then equal to the left hand side. Hence, Eq. (6) holds. The existence of the solution (7) has now been proved.

Next, we prove the uniqueness. That is, we would like to show that Eq. (7) can be derived from Eqs. (5) and (6). As is well known, the general solution to the consistent Eq. (6) is given by

$$y = C^+ b + (I - C^+ C) w_1, \tag{8}$$

where w_1 is an arbitrary vector of the appropriate dimension. Combining Eqs. (5) and (8) provides

$$g + C^T \lambda = C^+ b + (I - C^+ C) w_1. \tag{9}$$

Multiplying both sides of Eq. (9) by C on the left yields

$$Cg + CC^T \lambda = CC^+ b, \tag{10}$$

because $C(I - C^+ C) w_1 = (C - CC^+ C) w_1 = (C - C) w_1 = 0$. From Eq. (10), we see that

$$CC^T \lambda = CC^+ b - Cg. \tag{11}$$

The vector $C^T \lambda$ that satisfies the consistent Eq. (11) is then given by

$$C^T \lambda = C^+ (CC^+ b - Cg) + (I - C^+ C) w_2, \tag{12}$$

or

$$C^T \lambda = C^+ (b - Cg) + (I - C^+ C) w_2, \tag{13}$$

where w_2 is an arbitrary vector of the appropriate dimension. Multiplying both sides of Eq. (13) by $C^T (C^T)^+$ on the left gives

$$C^T (C^T)^+ C^T \lambda = C^T (C^T)^+ C^+ (b - Cg) + \left[C^T (C^T)^+ - C^T (C^T)^+ C^+ C\right] w_2. \tag{14}$$

Based on the properties of the generalized inverses, it holds that: $C^T (C^T)^+ C^T = C^T$, $C^T (C^T)^+ C^+ = C^+$, $(C^+ C)^T = C^+ C$, and $(C^T)^+ = (C^+)^T$. Therefore, Eq. (14) is simplified as

$$\begin{aligned} C^T \lambda &= C^+ (b - Cg) + [C^T (C^+)^T - C^+ C] w_2 \\ &= C^+ (b - Cg) + [(C^+ C)^T - C^+ C] w_2 \\ &= C^+ (b - Cg) + [C^+ C - C^+ C] w_2 \\ &= C^+ (b - Cg). \end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (5) then yields

$$y = g + C^+ (b - Cg). \tag{16}$$

Moreover, from Eq. (15), we see that the Lagrange multiplier λ can be written as

$$\lambda = (C^T)^+C^+(b - Cg) + [I - (C^T)^+(C^T)]w_3$$
$$= (C^T)^+C^+(b - Cg) + [I - CC^+]w_3, \tag{17}$$

where w_3 is an arbitrary vector of the appropriate dimension. This proof also reveals that although the Lagrange multiplier λ need not be unique for the set of linear algebraic Eqs. (1), the vector $C^T\lambda$, or equivalently, the vector $A^T\lambda$ is unique.

4. Alternative proof of Gauss' principle of least constraint

Gauss' principle states that the actual acceleration vector is the one that minimizes the expression

$$G = (\ddot{x} - a)^T M(\ddot{x} - a), \tag{18}$$

subject to

$$A\ddot{x} = b. \tag{19}$$

It is, of course, not possible to improve upon Gauss' simple and lucid proof (Gauss, 1829), which involves no more than the principle of virtual work and the law of cosines. But we see that if we put $z = M^{1/2}(\ddot{x} - a)$, then, Eqs. (18) and (19) are equivalent to

$$G = z^T z, \tag{20}$$

and

$$(AM^{-1/2})z = b - Aa. \tag{21}$$

Gauss' principle turns out to be finding the shortest solution of the consistent Eq. (21). The solution to it can be simply expressed by

$$z = (AM^{-1/2})^+(b - Aa). \tag{22}$$

or

$$M^{1/2}(\ddot{x} - a) = (AM^{-1/2})^+(b - Aa). \tag{23}$$

From Eq. (23), it is then clear that $\ddot{x} = a + M^{-1/2}(AM^{-1/2})^+(b - Aa)$. Since this is the GI form, the correct expression for the constrained acceleration that is equivalent to Lagrange's equations of motion of the first kind, the proof of Gauss' principle is completed.

In the earlier paper (Kalaba & Udwadia 1992), the GI form of the equations of motion has been proved to satisfy Gauss' principle of least constraint.

5. An application

To see the simplicity of the GI form of the equations of motion and its equivalence to Lagrange's equations of motion of the first kind and Gauss' principle, let us take a simple pendulum as an example.

Consider the trivial case in Figure 1 in which we want to determine the equations of motion for a material point with mass m and coordinates x_1, x_2.

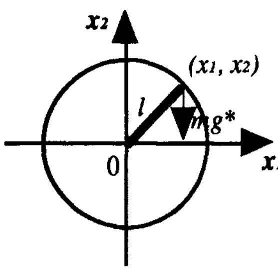

Fig. 1. The motion of a pendulum

The GI form suggests that we do the following. First, identify the free motion acceleration and the mass matrix. In this case, they are $\boldsymbol{a}_{2\times 1} = \begin{pmatrix} 0 \\ -g^* \end{pmatrix}$ and $\boldsymbol{M}_{2\times 2} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, where $-g^*$ is the acceleration due to gravity. Consequently, $\boldsymbol{M}_{2\times 2}{}^{-1/2} = \begin{pmatrix} m^{-1/2} & 0 \\ 0 & m^{-1/2} \end{pmatrix} = m^{-1/2}\boldsymbol{I}$. Secondly, write out all the constraint equations; in this case, the only constraint is $x_1^2 + x_2^2 = l^2$. Thirdly, get the linear restrictions on the acceleration vector $\ddot{\boldsymbol{x}}$, $\underset{2\times 1}{\ddot{\boldsymbol{x}}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}$, by two differentiations of the holonomic constraint. This results in the equation

$$x_1\ddot{x}_1 + x_2\ddot{x}_2 = -(\dot{x}_1^2 + \dot{x}_2^2). \tag{24}$$

In matrix form, Eq. (24) is

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -(\dot{x}_1^2 + \dot{x}_2^2), \tag{25}$$

which indicates that $\boldsymbol{A}_{1\times 2} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$ and $\boldsymbol{b}_{1\times 1} = -(\dot{x}_1^2 + \dot{x}_2^2)$. Since $\boldsymbol{AM}^{-1/2} = m^{-1/2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}$, the generalized inverse of this vector is simply $(\boldsymbol{AM}^{-\frac{1}{2}})^+ = \frac{1}{m^{-1/2}(x_1^2+x_2^2)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Lastly, substitute all the required values into Eq. (2). We obtain

$$\ddot{\boldsymbol{x}} = \boldsymbol{a} + \boldsymbol{M}^{-1/2}(\boldsymbol{A}\boldsymbol{M}^{-1/2})^{+}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{a})$$
$$= \begin{pmatrix} 0 \\ -g^* \end{pmatrix} + \frac{m^{-1/2}}{m^{-1/2}(x_1^2 + x_2^2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \left[-(\dot{x}_1^2 + \dot{x}_2^2) - (\, x_1 \quad x_2 \,) \begin{pmatrix} 0 \\ -g^* \end{pmatrix} \right]$$
$$= \begin{pmatrix} 0 \\ -g^* \end{pmatrix} + \frac{[x_2 g^* - (\dot{x}_1^2 + \dot{x}_2^2)]}{(x_1^2 + x_2^2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{26}$$

That is, the equations of motion for the pendulum are

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -g^* \end{pmatrix} + \frac{[x_2 g^* - (\dot{x}_1^2 + \dot{x}_2^2)]}{(x_1^2 + x_2^2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{27}$$

On the other hand, Lagrange's equations of motion of the first kind take the form of the equation

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -mg^* \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{28}$$

plus the constraint equation

$$x_1^2 + x_2^2 = l^2. \tag{29}$$

Differentiating Eq. (29) twice with respect to t produces

$$x_1 \ddot{x}_1 + x_2 \ddot{x}_2 = -(\dot{x}_1^2 + \dot{x}_2^2). \tag{30}$$

From Eq. (28), we see that

$$\ddot{x}_1 = \lambda x_1 / m, \tag{31}$$

and

$$\ddot{x}_2 = -g^* + \lambda x_2 / m. \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (30) yields

$$\lambda x_1^2 / m - x_2 g^* + \lambda x_2^2 / m = -(\dot{x}_1^2 + \dot{x}_2^2), \tag{33}$$

or

$$\lambda = \frac{[x_2 g^* - (\dot{x}_1^2 + \dot{x}_2^2)]m}{x_1^2 + x_2^2}. \tag{34}$$

Substituting Eq. (34) into Eqs. (31) and (32) then gives

$$\ddot{x}_1 = \frac{[x_2 g^* - (\dot{x}_1^2 + \dot{x}_2^2)]x_1}{x_1^2 + x_2^2} \tag{35}$$

and

$$\ddot{x}_2 = -g^* + \frac{[x_2 g^* - (\dot{x}_1^2 + \dot{x}_2^2)]x_2}{x_1^2 + x_2^2}, \tag{36}$$

respectively, which are identical with the results obtained from the GI form.

For comparison, let us now consider Gauss' principle of least constraint. Based on the discussions in Section 4, we know that the actual acceleration of the pendulum is the one that minimizes the value of Eq. (18) subject to Eq. (19). Specifically, Gauss' principle poses an optimization problem which minimizes

$$\begin{aligned} G &= (\ddot{\boldsymbol{x}} - \boldsymbol{a})^T \boldsymbol{M} (\ddot{\boldsymbol{x}} - \boldsymbol{a}) \\ &= (\ddot{x}_1 \quad \ddot{x}_2 + g^*) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 + g^* \end{pmatrix} \\ &= m[\ddot{x}_1^2 + (\ddot{x}_2 + g^*)^2], \end{aligned} \tag{37}$$

subject to

$$x_1\ddot{x}_1 + x_2\ddot{x}_2 = -(\dot{x}_1^2 + \dot{x}_2^2). \tag{38}$$

To transform this constrained optimization problem into an unconstrained one, we use the elimination method. Assuming $x_1 \neq 0$, Eq. (38) then gives

$$\ddot{x}_1 = -[(\dot{x}_1^2 + \dot{x}_2^2) + x_2\ddot{x}_2]/x_1. \tag{39}$$

Substituting Eq. (39) into Eq. (37) provides

$$G = m[(\dot{x}_1^2 + \dot{x}_2^2 + x_2\ddot{x}_2)^2/x_1^2 + (\ddot{x}_2 + g^*)^2]. \tag{40}$$

The minimizing value of $\ddot{x}_2$ is then obtained by setting $\partial G/\partial \ddot{x}_2 = 0$, i.e.,

$$(\dot{x}_1^2 + \dot{x}_2^2 + x_2\ddot{x}_2)x_2/x_1^2 + \ddot{x}_2 + g^* = 0. \tag{41}$$

Therefore,

$$\ddot{x}_2 = -\frac{x_1^2 g^* + (\dot{x}_1^2 + \dot{x}_2^2)x_2}{x_1^2 + x_2^2}. \tag{42}$$

Substituting Eq. (42) into Eq. (39) gives

$$\begin{aligned} \ddot{x}_1 &= -\frac{\dot{x}_1^2 + \dot{x}_2^2}{x_1} + \frac{x_2}{x_1}\left[\frac{x_1^2 g^* + (\dot{x}_1^2 + \dot{x}_2^2)x_2}{x_1^2 + x_2^2}\right] \\ &= \frac{\left[x_2 g^* - (\dot{x}_1^2 + \dot{x}_2^2)\right] x_1}{x_1^2 + x_2^2}. \end{aligned} \tag{43}$$

It is easy to verify that Eqs. (43) and (42) are equivalent to Eqs. (35) and (36), respectively. Notice that these equations hold even if $x_1 = 0$.

6. Conclusions

The occurrence of the generalized inverse of the matrix $AM^{-1/2}$ is a cause of some consternation. But this should not be, because whatever difficulty is involved in integrating the GI formula, the same difficulty is encountered in using the equations of motion of the first kind. This is because they are identical, but expressed in different forms. The explicit representation given by the GI formula exposes, but does not cover up, any difficulties hidden in the employment of Lagrange's equations. The power of the GI formulation is hinted at in the simple proof of Gauss' principle of least constraint. Moreover, the transformation $z = M^{1/2}(\ddot{x} - a)$ suggests that in the original $3n$ dimensional space, distances should be multiplied by the square roots of masses. Then Gauss' principle assumes the simplest form that we seek the shortest length solution of the system $(AM^{-1/2})z = b - Aa$. Future communications will deal with the role of the GI formulation in Lagrange's equations of motion of the second kind, in Hamiltonian mechanics, in the Gibbs-Appell approach and in that of Dirac. Questioning the GI formula is no better than tilting with windmills.

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